

# ON THE EXPONENT OF SPINOR GROUPS

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## 1. INTRODUCTION

Let  $G$  be a split simple simply connected group of rank  $n$  over a field  $F$ . Fix a maximal split torus  $T$  of  $G$  and a Borel subgroup  $B$  containing  $T$ . We denote by  $W$  the Weyl group of  $G$  with respect to  $T$ . Let  $\Lambda$  be the weight lattice of  $G$  (hence,  $T^* = \Lambda$ ).

We denote by  $\omega_1, \dots, \omega_n$  the fundamental weights of  $\Lambda$ . We let  $I_K := \text{Ker}(\mathbb{Z}[\Lambda] \rightarrow \mathbb{Z})$  and  $I_{CH} := \text{Ker}(S^*(\Lambda) \rightarrow \mathbb{Z})$  be the augmentation ideals, where  $\mathbb{Z}[\Lambda] \rightarrow \mathbb{Z}$  (respectively,  $S^*(\Lambda) \rightarrow \mathbb{Z}$ ) is the map from the group ring  $\mathbb{Z}[\Lambda]$  (respectively, the symmetric algebra) of  $\Lambda$  to the ring of integers by sending  $e^\lambda$  to 1 (respectively, any element of positive degree to 0).

For any  $i \geq 0$ , we consider the ring homomorphism

$$\phi^{(i)} : \mathbb{Z}[\Lambda] \rightarrow \mathbb{Z}[\Lambda]/I_K^{i+1} \rightarrow S^*(\Lambda)/I_{CH}^{i+1} \rightarrow S^i(\Lambda),$$

where the first and the last maps are projections and the middle map sends  $e^{\sum_{j=1}^n a_j \omega_j}$  to  $\prod_{j=1}^n (1 - \omega_j)^{-a_j}$ . The  $i$ th-exponent of  $G$  (denoted by  $\tau_i$ ), as introduced in [1], is the gcd of all nonnegative integers  $N_i$  satisfying

$$N_i \cdot (I_{CH}^W)^{(i)} \subseteq \phi^{(i)}(I_K^W),$$

where  $I_K^W := \langle \mathbb{Z}[\Lambda]^W \cap I_K \rangle$  (respectively,  $I_{CH}^W := \langle S^*(\Lambda)^W \cap I_{CH} \rangle$ ) denotes the  $W$ -invariant augmentation ideal of  $\mathbb{Z}[\Lambda]$  (respectively,  $S^*(\Lambda)$ ) and  $(I_{CH}^W)^{(i)} = I_{CH}^W \cap S^i(\Lambda)$ . Informally, these numbers  $\tau_i$  measure how far is the ring  $S^*(\Lambda)^W$  from being a polynomial ring in basic invariants.

For any  $i \leq 4$ , it was shown that the  $i$ th-exponent of  $G$  divides the Dynkin index in [1] and this integer was used to estimate the torsion of the Grothendieck gamma filtration and the Chow groups of  $E/B$ , where  $E/B$  denotes the twisted form of the variety of Borel subgroups  $G/B$  for a  $G$ -torsor  $E$ .

In this paper, we show that all the remaining exponents of spinor groups divide the Dynkin index 2.

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## 2. EXPONENT

Let  $G$  be  $\mathbf{Spin}_{2n+1}$  ( $n \geq 3$ ) or  $\mathbf{Spin}_{2n}$  ( $n \geq 4$ ). The fundamental weights are defined by

$$\begin{aligned} \omega_1 = e_1, \omega_2 = e_1 + e_2, \dots, \omega_{n-1} = e_1 + \dots + e_{n-1}, \omega_n = \frac{e_1 + \dots + e_n}{2}, \\ \omega_1 = e_1, \omega_2 = e_1 + e_2, \dots, \omega_{n-1} = \frac{e_1 + \dots + e_{n-1} - e_n}{2}, \omega_n = \frac{e_1 + \dots + e_n}{2}, \end{aligned}$$

respectively, where the canonical basis of  $\mathbb{R}^n$  is denoted by  $e_i$  ( $1 \leq i \leq n$ ).

For  $1 \leq i \leq n$ , let

$$(1) \quad q_{2i} := e_1^{2i} + \dots + e_n^{2i}$$

be the basic invariants of the group  $G$ , i.e., be algebraically independent homogeneous generators of  $S^*(\Lambda)^W$  as a  $\mathbb{Q}$ -algebra (see [2, §3.5 and §3.12]), together with

$$(2) \quad q'_n := e_1 \cdots e_n$$

if  $G = \mathbf{Spin}_{2n}$ .

For any  $\lambda \in \Lambda$ , we denote by  $W(\lambda)$  the  $W$ -orbit of  $\lambda$ . For any finite set  $A$  of weights, we denote  $-A$  the set of opposite weights.

The Weyl groups of  $\mathbf{Spin}_{2n+1}$  and  $\mathbf{Spin}_{2n}$  are  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$  and  $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ , respectively. Hence, by the action of these Weyl groups, one has the following decomposition of  $W$ -orbits: if  $G = \mathbf{Spin}_{2n+1}$  (respectively,  $G = \mathbf{Spin}_{2n}$ ), then for any  $1 \leq k \leq n-1$  (respectively,  $1 \leq k \leq n-2$ )

$$(3) \quad W(\omega_k) = W_+(\omega_k) \cup -W_+(\omega_k),$$

where  $W_+(\omega_k) = \{e_{i_1} \pm \dots \pm e_{i_k} \mid i_1 < \dots < i_k\}$ . If  $n$  is even, then the  $W$ -orbits of the last two fundamental weights of  $\mathbf{Spin}_{2n}$  are given by

$$(4) \quad W(\omega_{n-1}) = W_+(\omega_{n-1}) \cup -W_+(\omega_{n-1}) \text{ and } W(\omega_n) = W_+(\omega_n) \cup -W_+(\omega_n),$$

where  $W_+(\omega_{n-1})$  (respectively,  $W_+(\omega_n)$ ) is the subset of  $W(\omega_{n-1})$  (respectively,  $W(\omega_n)$ ) containing elements of the positive sign of  $e_1$ .

For any  $\lambda = \sum_{j=1}^n a_j \omega_j \in \Lambda$  and any integer  $m \geq 0$ , we set  $\lambda(m) = \sum_{j=1}^n a_j \omega_j^m$ . For example,  $\lambda(0) = \sum_{j=1}^n a_j$  and  $\lambda(1) = \lambda$ . We shall need the following lemma:

**Lemma 2.1.** (i) If  $G$  is  $\mathbf{Spin}_{2n+1}$  (respectively,  $\mathbf{Spin}_{2n}$ ), then for any odd integer  $p$ , any nonnegative integers  $m_1, \dots, m_p$  and, any  $1 \leq k \leq n-1$  (respectively, any  $1 \leq k \leq n-2$ ), we have

$$\sum_{\lambda \in W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) = 0.$$

(ii) If  $G$  is  $\mathbf{Spin}_{2n}$  with odd  $n$ , then for any even integer  $p$  and any nonnegative integers  $m_1, \dots, m_p$ , we have

$$\sum_{\lambda \in W(\omega_n)} \lambda(m_1) \cdots \lambda(m_p) = \sum_{\lambda \in W(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p).$$

(iii) If  $G$  is  $\mathbf{Spin}_{2n}$ , then for any odd integer  $p < n$  and any nonnegative integers  $m_1, \dots, m_p$ , we have

$$\sum_{\lambda \in W(\omega_n)} \lambda(m_1) \cdots \lambda(m_p) = \sum_{\lambda \in W(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p) = 0.$$

*Proof.* (i) It follows from (3) that

$$\begin{aligned} \sum_{\lambda \in W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) &= \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) + \sum_{\lambda \in -W_+(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) \\ &= \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) - \sum_{\lambda \in W_+(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) \\ &= 0. \end{aligned}$$

(ii) If  $G$  is  $\mathbf{Spin}_{2n}$  with odd  $n$ , then we have  $W(\omega_n) = -W(\omega_{n-1})$ . Hence, the result immediately follows from the assumption that  $p$  is even.

(iii) If  $n$  is even, then the result follows from (4) by the same argument as in the proof of (i). In general, note that for any  $\lambda_{i_1}, \dots, \lambda_{i_p} \in W_+(\omega_1)$  the term  $\lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)/2^p$  (respectively,  $-\lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)/2^p$ ) appears  $2^{n-2}$  times (respectively,  $2^{n-2}$ ) in both sums in (iii).  $\square$

Let  $p$  be an even integer and  $q \geq 2$  an integer. For any nonnegative integers  $m_1, \dots, m_p$ , we define

$$\Lambda(p, q)(m_1, \dots, m_p) := \sum \lambda_{j_1}(m_1) \cdots \lambda_{j_p}(m_p),$$

where the sum ranges over all different  $\lambda_{i_1}, \dots, \lambda_{i_q} \in W_+(\omega_1)$  and all  $\lambda_{i_1}, \dots, \lambda_{i_p} \in \{\lambda_{i_1}, \dots, \lambda_{i_q}\}$  such that the numbers of  $\lambda_{i_1}, \dots, \lambda_{i_q}$  appearing in  $\lambda_{i_1}, \dots, \lambda_{i_p}$  are all nonnegative even solutions of  $x_1 + \dots + x_q = p$ . If  $p < 2q$ , then we set  $\Lambda(p, q)(m_1, \dots, m_p) = 0$ . Given  $m_1, \dots, m_p$ , we simply write  $\Lambda(p, q)$  for  $\Lambda(p, q)(m_1, \dots, m_p)$ .

For instance,  $\Lambda(4, 2)$  is the sum of  $\lambda_{j_1}(m_1)\lambda_{j_2}(m_2)\lambda_{j_3}(m_3)\lambda_{j_4}(m_4)$  for all  $j_1, j_2, j_3, j_4 \in \{i, j\}$  and all  $1 \leq i \neq j \leq n$  such that two  $i$ 's and two  $j$ 's appear in  $j_1, j_2, j_3, j_4$ .

**Example 2.2.** We observe that

$$(5) \quad (x_1 + x_2)(x'_1 + x'_2) + (x_1 - x_2)(x'_1 - x'_2) = 2(x_1x'_1 + x_2x'_2).$$

If  $G$  is  $\mathbf{Spin}_{2n+1}$  or  $\mathbf{Spin}_{2n}$ , then by (3) and (5) we have

$$\sum_{W_+(\omega_2)} \lambda(m_1)\lambda(m_2) = 2(n-1) \sum_{W_+(\omega_1)} \lambda(m_1)\lambda(m_2)$$

for any nonnegative integers  $m_1$  and  $m_2$  as we have  $(n-1)$  choices of such pairs in the left hand side of (5) from  $W_+(\omega_2)$ , which implies that

$$\sum_{W(\omega_2)} \lambda(m_1)\lambda(m_2) = 2(n-1) \sum_{W(\omega_1)} \lambda(m_1) \cdots \lambda(m_2),$$

(cf. [1, Lemma 5.1(ii)]). For any even  $p \geq 4$ , we apply the same argument with the expansion of  $(x_1 + x_2) \cdots (x_1^{(p)} + x_2^{(p)}) + (x_1 - x_2) \cdots (x_1^{(p)} - x_2^{(p)})$ . Then, we have

$$\sum_{W_+(\omega_2)} \lambda(m_1) \cdots \lambda(m_p) = 2(n-1) \sum_{W_+(\omega_1)} \lambda(m_1) \cdots \lambda(m_p) + 2\Lambda(p, 2),$$

which implies that

$$\sum_{W(\omega_2)} \lambda(m_1) \cdots \lambda(m_p) = 2(n-1) \sum_{W(\omega_1)} \lambda(m_1) \cdots \lambda(m_p) + 2^2 \Lambda(p, 2).$$

We generalize Example 2.2 to any  $\omega_k$  as follows.

**Lemma 2.3.** *If  $G$  is  $\mathbf{Spin}_{2n+1}$  (respectively,  $\mathbf{Spin}_{2n}$ ), then for any  $1 \leq k \leq n-1$  (respectively,  $1 \leq k \leq n-2$ ), any even  $p$ , and any nonnegative integers  $m_1, \dots, m_p$  we have*

$$\sum_{W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p) = 2^{k-1} \binom{n-1}{k-1} \sum_{W(\omega_1)} \lambda(m_1) \cdots \lambda(m_p) + \sum_{j=2}^k 2^k \binom{n-j}{k-j} \Lambda(p, j).$$

*Proof.* For any  $\lambda \in W(\omega_1)$ , there are  $2^k \binom{n-1}{k-1}$  choices of the element containing  $\lambda$  in  $W(\omega_k)$ , thus we have the term  $2^{k-1} \binom{n-1}{k-1} \sum_{W(\omega_1)} \lambda(m_1) \cdots \lambda(m_p)$  in  $\sum_{W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p)$ .

If an element  $\lambda \in W(\omega_1)$  appears odd times in a term  $\lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)$  of  $\sum_{W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p)$ , where  $\lambda_{i_1}, \dots, \lambda_{i_p} \in W(\omega_1)$ , then by the action of Weyl group this term vanishes in  $\sum_{W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p)$ . Hence, the remaining terms in  $\sum_{W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p)$  are a linear combination of  $\Lambda(p, j)$  for all  $2 \leq j \leq k$  such that  $p \geq 2k$ . As each term  $\Lambda(p, j)$  appears  $2^k \binom{n-j}{k-j}$  times in  $\sum_{W(\omega_k)} \lambda(m_1) \cdots \lambda(m_p)$ , the result follows.  $\square$

For any  $\lambda \in \Lambda$ , we denote by  $\rho(\lambda)$  the sum of all elements  $e^\mu \in \mathbb{Z}[\Lambda]$  over all elements  $\mu$  of  $W(\lambda)$ . Let  $i! \cdot \phi^{(i)}(e^\lambda) = \lambda^i + S_i$  for any  $i \geq 1$ , where  $S_i$  is the sum of remaining terms in  $i! \cdot \phi^{(i)}(e^\lambda)$  and  $\lambda = \sum a_j \omega_j$ ,  $a_j \in \mathbb{Z}$ . Hence, for any fundamental weight  $\omega_k$  we have

$$(6) \quad i! \cdot \phi^{(i)}(\rho(\omega_k)) = \sum_{W(\omega_k)} \lambda^i + \sum_{W(\omega_k)} S_i.$$

We view  $i! \cdot \phi^{(i)}(e^\lambda)$  as a polynomial in variables  $\lambda, \lambda(m_1), \dots, \lambda(m_j)$  for some nonnegative integers  $m_1, \dots, m_j$ . Let  $T_i$  be the sum of monomials in  $S_i$  whose degrees are even.

If  $G$  is  $\mathbf{Spin}_{2n+1}$  (respectively,  $\mathbf{Spin}_{2n}$ ), then by Lemma 2.1(i) the equation (6) reduces to

$$(7) \quad i! \cdot \phi^{(i)}(\rho(\omega_k)) = \sum_{W(\omega_k)} \lambda^i + \sum_{W(\omega_k)} T_i.$$

for any  $1 \leq k \leq n-1$  (respectively  $1 \leq k \leq n-2$ ).

Given  $p$  and  $q$ , we define

$$\Omega(p, q) := \sum \Lambda(p, q)(m_1, \dots, m_p),$$

where the sum ranges over all  $m_1, \dots, m_p$  which appear in all monomials of  $T_i$ .

**Example 2.4.** (i) If  $G$  is  $\mathbf{Spin}_{2n+1}$  or  $\mathbf{Spin}_{2n}$  and  $i = 4$ , then by (7) and Lemma 2.3 we have

$$\begin{aligned} 4!\phi^{(4)}(\rho(\omega_1)) &= \sum_{W(\omega_1)} \lambda^4 + \sum_{W(\omega_1)} T_4, \\ 4!\phi^{(4)}(\rho(\omega_2)) &= \sum_{W(\omega_2)} \lambda^4 + \sum_{W(\omega_2)} T_4 \\ &= \sum_{W(\omega_2)} \lambda^4 + 2(n-1) \sum_{W(\omega_1)} T_4, \end{aligned}$$

which implies that

$$4!(\phi^{(4)}(\rho(\omega_2)) - 2(n-1)\phi^{(4)}(\rho(\omega_1))) = \sum_{W(\omega_2)} \lambda^4 - 2(n-1) \sum_{W(\omega_1)} \lambda^4.$$

By Lemma 2.3, the right-hand side of the above equation is equal to

$$4\Lambda(4, 2) = 4 \cdot \frac{4!}{2!2!} \sum_{i < j} e_i^2 e_j^2.$$

Hence, we have

$$\phi^{(4)}(\rho(\omega_2)) - 2(n-1)\phi^{(4)}(\rho(\omega_1)) = \sum_{i < j} e_i^2 e_j^2.$$

(ii) If  $G$  is  $\mathbf{Spin}_{2n+1}$  ( $n \geq 4$ ) or  $\mathbf{Spin}_{2n}$  ( $n \geq 5$ ) and  $i = 6$ , then by (7) and Lemma 2.3 we have

$$\begin{aligned} 6!\phi^{(6)}(\rho(\omega_1)) &= \sum_{W(\omega_1)} \lambda^6 + \sum_{W(\omega_1)} T_6, \\ 6!\phi^{(6)}(\rho(\omega_2)) &= \sum_{W(\omega_2)} \lambda^6 + 2(n-1) \sum_{W(\omega_1)} T_6 + 4\Omega(4, 2), \\ 6!\phi^{(6)}(\rho(\omega_3)) &= \sum_{W(\omega_3)} \lambda^6 + 4 \binom{n-1}{2} \sum_{W(\omega_1)} T_6 + 8(n-2)\Omega(4, 2), \end{aligned}$$

which implies that

$$\phi^{(6)}(\rho(\omega_3)) - 2(n-2)\phi^{(6)}(\rho(\omega_2)) + 2(n-1)(n-2)\phi^{(6)}(\rho(\omega_1)) = \sum_{i < j < k} e_i^2 e_j^2 e_k^2.$$

**Lemma 2.5.** (i) If  $G$  is  $\mathbf{Spin}_{2n}$ , then we have

$$\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n = n!e_1 \cdots e_n.$$

(ii) If  $G$  is  $\mathbf{Spin}_{2n}$ , then for any  $1 \leq p \leq n-1$  and any nonnegative integers  $m_1, \dots, m_p$  we have

$$\sum_{W(\omega_n)} \lambda(m_1) \cdots \lambda(m_p) = \sum_{W(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p).$$

*Proof.* (i) First, assume that  $n \geq 4$  is even. We show that

$$\sum_{W_+(\omega_n)} \lambda^n - \sum_{W_+(\omega_{n-1})} \lambda^n = (n!/2)e_1 \cdots e_n.$$

As  $|W_+(\omega_n)| = |W_+(\omega_{n-1})| = 2^{n-2}$ , we have

$$(n!/2^n)2^{n-2}e_1 \cdots e_n - (-(n!/2^n)2^{n-2}e_1 \cdots e_n) = (n!/2)e_1 \cdots e_n$$

in  $\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n$ . If one of the exponents  $i_1, \dots, i_n$  in  $e_1^{i_1} \cdots e_n^{i_n}$  (except the case  $i_1 = \cdots = i_n = 1$ ) is odd, then from the orbits  $W_+(\omega_n)$  and  $W_+(\omega_{n-1})$  this monomial vanishes in each sum of  $\sum_{W_+(\omega_n)} \lambda^n - \sum_{W_+(\omega_{n-1})} \lambda^n$ . Otherwise, the terms  $2^{n-2} \sum_{j=1}^n e_j^n, \Lambda(n, 2) \cdots, \Lambda(n, n/2)$  with  $m_1 = \cdots = m_n = 1$  are in both  $\sum_{W_+(\omega_n)} \lambda^n$  and  $\sum_{W_+(\omega_{n-1})} \lambda^n$ .

Now, we assume that  $n \geq 4$  is odd. As  $|W(\omega_n)| = |W(\omega_{n-1})| = 2^{n-1}$ , we have

$$(n!/2^n)2^{n-1}e_1 \cdots e_n - (-(n!/2^n)2^{n-1}e_1 \cdots e_n) = n!e_1 \cdots e_n$$

in  $\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n$ . By the same argument, if one of the exponents  $i_1, \dots, i_n$  in  $e_1^{i_1} \cdots e_n^{i_n}$  (except the case  $i_1 = \cdots = i_n = 1$ ) is odd, then this monomial vanishes in each sum of  $\sum_{W(\omega_n)} \lambda^n - \sum_{W(\omega_{n-1})} \lambda^n$ . This completes the proof of (i).

(ii) By Lemma 2.1(ii)(iii), it is enough to consider the case where both  $n$  and  $p$  are even. For any  $p$  and any  $n \geq p+2$ , we have  $2^{n-2}(\sum_{W_+(\omega_1)} \lambda(m_1) \cdots \lambda(m_p))$  in both  $\sum_{W_+(\omega_n)} \lambda(m_1) \cdots \lambda(m_p)$  and  $\sum_{W_+(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p)$ . By the action of Weyl group, any term  $\lambda_{i_1}(m_1) \cdots \lambda_{i_p}(m_p)$ , where an element  $\lambda \in W(\omega_1)$  appears odd times in either  $\sum_{W_+(\omega_n)} \lambda(m_1) \cdots \lambda(m_p) - 2^{n-2}(\sum_{W_+(\omega_1)} \lambda(m_1) \cdots \lambda(m_p))$  or  $\sum_{W_+(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p) - 2^{n-2}(\sum_{W_+(\omega_1)} \lambda(m_1) \cdots \lambda(m_p))$ , vanishes. As each term of  $\Lambda(p, 2), \dots, \Lambda(p, p/2)$  appears in both  $\sum_{W_+(\omega_n)} \lambda(m_1) \cdots \lambda(m_p)$  and  $\sum_{W_+(\omega_{n-1})} \lambda(m_1) \cdots \lambda(m_p)$ , this completes the proof.  $\square$

**Theorem 2.6.** *If  $G$  is  $\mathbf{Spin}_{2n+1}$  (respectively,  $\mathbf{Spin}_{2n}$ ), then for any  $i \geq 3$  and any  $n \geq [i/2] + 1$  (respectively,  $n \geq [i/2] + 2$ ) the exponent  $\tau_i$  divides the Dynkin index  $\tau_2 = 2$ .*

*Proof.* As  $B_2 = C_2$  and  $D_3 = A_3$ , we have  $1 = \tau_3 \mid 2$  by [1, Theorem 5.4]. If  $G$  is  $\mathbf{Spin}_{2n}$  for any  $n \geq 4$ , then by Lemma 2.5(i)(ii) we have

$$q'_n = \phi^{(n)}(\rho(\omega_n)) - \phi^{(n)}(\rho(\omega_{n-1})),$$

which implies that the invariant  $q'_n$  is in the ideal generated by the image of  $\phi^{(n)}$ . As there are no invariants of odd degree except  $q'_n$ , we have

$$\tau_{2i+1} \mid \tau_{2i}$$

for all  $i \geq 1$ . Therefore, it suffices to show that  $\tau_{2i} \mid \tau_2$  for any  $i \geq 2$ .

By Lemma 2.3 together with the same argument as in Example 2.4 we have

$$(8) \quad \phi^{(2i)}(\rho(\omega_i)) + \sum_{j=1}^{i-1} a_j \phi^{(2i)}(\rho(\omega_{i-j})) = \sum_{j_1 < \dots < j_i} e_{j_1}^2 \cdots e_{j_i}^2,$$

where the integers  $a_1, \dots, a_{i-1}$  satisfy

$$\left( \sum_{j=k}^{i-2} 2^{j+1} \binom{n-1-k}{j-k} a_{j+1} \right) + 2^i \binom{n-1-k}{i-1-k} = 0,$$

for  $0 \leq k \leq i-2$ . Let  $p_i$  be the right-hand side of (8). Then this equation implies that  $p_i$  is in the image of  $\phi^{(2i)}$ .

We show that the invariant  $q_{2i}$  is in the ideal  $\phi^{(2i)}(I_K^W)$  for any  $i \geq 2$ . We proceed by induction on  $i$ . As  $q_2 = \phi^{(2)}(\rho(\omega_1))$ , the case  $i = 2$  is obvious. By Newton's identities we have

$$(9) \quad (-1)^{i-1} q_{2i} = i p_i - \sum_{j=1}^{i-1} (-1)^{j-1} p_{i-1-j} q_{2j}$$

with  $p_0 = 1$ . By the induction hypothesis, the sum of (9) is in  $\phi^{(2i)}(I_K^W)$ . Hence,  $q_{2i}$  is in  $\phi^{(2i)}(I_K^W)$ .  $\square$

For any nonnegative integer  $n$ , we denote by  $v_2(n)$  the 2-adic valuation of  $n$ . For a smooth projective variety  $X$  over  $F$ , we denote by  $\Gamma^* K(X)$  the gamma filtration on the Grothendieck ring  $K(X)$ . We let  $c_{CH} : S^*(\Lambda) \rightarrow CH(G/B)$  be the characteristic map.

**Corollary 2.7.** *Let  $G$  be  $\mathbf{Spin}_{2n}$  (respectively,  $\mathbf{Spin}_{2n+1}$ ). If  $2^{m(i)}(\ker c_{CH})^{(i)} \subseteq (I_{CH}^W)^{(i)}$  for some nonnegative integer  $m(i)$ , then for any  $i \geq 3$  and any  $n \geq [i/2] + 2$  (respectively,  $n \geq [i/2] + 1$ ) the torsion of  $\Gamma^i K(G/B)/\Gamma^{i+1} K(G/B)$  is annihilated by  $2^{g(i)}$ , where  $g(i) = 1 + m(i) + v_2((i-1)!)$ .*

**Remark 2.8.** It is shown that  $m(3) = 0$  and  $m(4) = 1$  in [1, Lemma 6.4].

*Proof.* The proof of [1, Theorem 6.5] still works with Theorem 2.6.  $\square$

**Corollary 2.9.** *Let  $G$  be  $\mathbf{Spin}_{2n}$  (respectively,  $\mathbf{Spin}_{2n+1}$ ). If  $2^{m(i)}(\ker c_{CH})^{(i)} \subseteq (I_{CH}^W)^{(i)}$  for some nonnegative integer  $m(i)$ , then for any  $G$ -torsor  $E$ , any  $i \geq 3$  and any  $n \geq [i/2] + 2$  (respectively,  $n \geq [i/2] + 1$ ) the torsion of  $\mathrm{CH}^i(E/B)$  is annihilated by  $2^{t(i)}$ , where  $t(i) = 1 + \sum_{j=3}^i g(j) + v_2((i-1)!)$ .*

*Proof.* By [3, Theorem 2.2(2)], we have

$$\Gamma^i K(G/B)/\Gamma^{i+1} K(G/B) \simeq \Gamma^i K(E/B)/\Gamma^{i+1} K(E/B).$$

As the torsion of  $\mathrm{CH}^i(E/B)$  is annihilated by

$$(i-1)! \prod_{j=1}^i e(\Gamma^j K(E/B)/\Gamma^{j+1} K(E/B)),$$

where  $e(\Gamma^i K(E/B)/\Gamma^{i+1} K(E/B))$  denotes the finite exponent of its torsion subgroup (see [1, p.149]), the result follows from Corollary 2.7.  $\square$

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